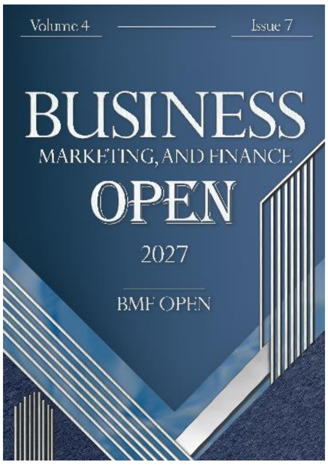
Optimal system design and optimal budget models in the presence of interval data

Mohammad Pouryusef [1], Ghasem Tohidi [2,*], Masoud Sanei [3], Farhad Hosseinzadeh Lotfi [4] and Shabnam Razavyan [5]



[1] Department of Mathematics, CT.C., Islamic Azad University, Tehran, Iran;
[2] Department of Mathematics, CT.C., Islamic Azad University, Tehran, Iran;
[3] Department of Mathematics, CT.C., Islamic Azad University, Tehran, Iran;
[4] Department of Science, SR.C., Islamic Azad University, Tehran, Iran;
[5] Department of Mathematics, ST.C., Islamic Azad University, Tehran, Iran;

* Correspondence: 2971011917@iau.ir

Citation: Pouryusef, M., Tohidi, G., Sanei, M., Hosseinzadeh Lotfi, F., & Razavyan, S. (2027). Optimal system design and optimal budget models in the presence of interval data . *Business, Marketing, and Finance Open*, 4(7), 1-30.

Received: 27 May 2026
Revised: 24 August 2026
Accepted: 04 September 2026
Initial Publication: 06 September 2026
Final Publication: 01 December 2027

Abstract: This paper develops an interval data envelopment analysis (DEA) framework for optimal system design (OSD) and optimal budget determination. Classical OSD DEA models generally assume that input and output data are known precisely. In many applications, however, production data are affected by measurement error, incomplete records, forecasting uncertainty, or managerial judgment and are therefore more realistically represented by lower and upper bounds. To address this issue, the present study extends profit-maximizing OSD DEA models to the case of interval input and output data. The proposed formulation enables a decision-making unit (DMU) to determine an economically desirable input-output configuration under a given budget when benchmark data are imprecise. The study also develops a complementary procedure for determining the optimal budget and for identifying budget congestion in the interval-data setting. In this framework, budget congestion occurs when forcing a DMU to consume a budget above the optimal level reduces profit rather than improving performance. The models are formulated as deterministic linear programming problems and can therefore be solved using standard optimization software. A numerical example with ten DMUs, three inputs, and two outputs is used to illustrate the applicability of the proposed approach. The results show that the interval OSD framework provides a practical decision-support tool for profit-oriented resource allocation when production data are not exactly known.

Keywords: data envelopment analysis, optimal system design, resource allocation, interval data, optimal budget, budget congestion, parametric linear programming.

1. Introduction

Data envelopment analysis (DEA) has become one of the most influential non-parametric methodologies for evaluating the relative performance of decision-making units (DMUs) operating with multiple inputs and outputs. Its principal advantage lies in its capacity to construct an empirical production frontier without requiring a predetermined functional form for the underlying production technology. Within this framework, the observed performance of each DMU is compared with feasible combinations of peer units, thereby providing a basis for identifying efficient operations and determining potential improvements for inefficient units. Although the earliest applications of DEA concentrated primarily on technical efficiency, subsequent methodological developments increasingly incorporated economic dimensions of performance, including cost, revenue, profitability, resource allocation, and target setting. Directional distance functions played an important role in this transition by providing a flexible framework in which improvements in outputs and reductions in inputs can be considered simultaneously and in economically meaningful directions [1]. Consequently, DEA has evolved from a retrospective performance-evaluation technique into a broader decision-support methodology capable of addressing complex managerial questions concerning resource use, production planning, and economic optimization.

The distinction between technical efficiency and economic efficiency is particularly important in this context. A DMU may operate on, or close to, the technically efficient frontier while nevertheless employing an economically undesirable input-output configuration. Profit-oriented organizations are ultimately concerned not only with whether resources are transformed efficiently into outputs, but also with whether the chosen combination of inputs and outputs is economically appropriate given input costs, output prices, and resource limitations. Research on profitability within DEA has demonstrated that observed profitability can be decomposed into technically and allocatively meaningful components, thereby permitting a more informative evaluation of economic performance than conventional technical-efficiency measures alone [2]. Subsequent developments have constructed decomposable measures of profit efficiency and clarified the manner in which DEA can identify economically meaningful improvement targets rather than merely radial efficiency adjustments [3]. Likewise, profit-efficiency analysis based on shadow-price concepts has demonstrated that economically relevant benchmarking remains possible even when conventional market-price information is limited or cannot be directly assigned to all production units [4]. More recently, inverse DEA formulations have further extended economic DEA analysis by considering cost and revenue efficiency in settings where explicit price information is absent, illustrating the continuing effort to connect frontier analysis with realistic economic decision-making problems [5].

An important consequence of this development is a shift from asking how efficiently an existing system performs toward asking how an improved system should be designed. Conventional DEA ordinarily treats the observed input-output vectors of DMUs as given and evaluates their performance relative to a production possibility set generated from peer observations. However, managers frequently face prospective rather than merely evaluative decisions. They may need to decide how much of each input should be employed, what level of each output should be targeted, how available financial resources should be allocated, and what production configuration would maximize the economic value generated by a limited budget. These questions are central to optimal system design (OSD). DEA-based OSD approaches exploit information contained in the observed production technology to construct a feasible input-output combination that represents an economically desirable system rather than simply projecting an existing inefficient unit toward the frontier.

The development of DEA-type models for optimal system design provided a systematic mathematical basis for this prospective use of frontier analysis. Wei and Chang demonstrated that DEA technology can be employed not only to evaluate existing production systems but also to design optimal systems and determine economically appropriate budgets [6]. Under such an approach, the target system is constructed from the empirical production possibility set, while economic information such as input costs, output prices, and a total budget is incorporated directly into the optimization problem. The resulting model determines a feasible combination of inputs and outputs that is benchmarked against observed DMUs and is optimal according to a specified economic criterion. In the profit-maximizing formulation, the OSD problem directly identifies the production configuration that yields the greatest attainable difference between output revenue and input cost under the available budget [7]. This represents a substantive conceptual expansion of DEA because the analytical object is no longer restricted to the performance score of an existing DMU; instead, the methodology determines how the system itself should be configured.

The budget component of optimal system design is especially consequential because financial resources often constitute a binding constraint on organizational production possibilities. In many applications, decision makers are provided with a predetermined budget and are expected to transform that financial resource into an appropriate portfolio of inputs and outputs. Linear programming offers a natural mathematical structure for such problems because resource restrictions, technological feasibility conditions, and economic objectives can be represented simultaneously through linear constraints and objective functions [8]. Moreover, when the budget enters the optimization model parametrically, changes in the optimal objective value can be examined as the available budget varies. This allows decision makers to move beyond solving a production problem for one arbitrarily specified budget and instead investigate the relationship between financial resources and attainable economic performance.

Research on optimal budgets has shown that a larger available budget does not necessarily imply proportionately greater economic performance. Models specifically developed for system design and optimal budgeting indicate that the optimal response of a DEA-based production system may change across different ranges of the budget parameter [6]. Extensions to more complex series-network settings similarly demonstrate the importance of determining the budget level that is consistent with optimal system performance rather than assuming that all additional financial resources necessarily generate additional value [9]. From a managerial standpoint, this issue is fundamental. An organization that allocates financial resources beyond the level required to achieve its maximum attainable economic performance may create excess expenditure without obtaining a corresponding improvement in profit. Consequently, identifying both the optimal system and the economically appropriate budget can provide more useful decision support than maximizing resource utilization in isolation.

Despite these methodological advances, conventional OSD DEA formulations commonly rely on a restrictive assumption: the input and output observations defining the reference technology are treated as precisely known. In practical settings, however, production data are often imprecise. Input consumption may be affected by measurement errors, accounting approximations, incomplete records, reporting delays, or uncertain future requirements. Likewise, output quantities may depend on demand variability, forecasting errors, operational volatility, or managerial estimates. When the analysis concerns prospective planning, the problem becomes even more pronounced because the values used to construct benchmarks may represent expectations rather than fully observed realizations. Representing uncertain quantities as exact point estimates can therefore create an artificial degree of precision and may produce optimal designs that are sensitive to small changes in the underlying observations.

Interval data provide an attractive way to represent such uncertainty when the precise probability distribution of uncertain observations is unknown or unavailable. Instead of assigning a single value to an input or output, an interval specifies lower and upper bounds within which the quantity is believed to lie. This representation is comparatively simple, requires less information than probabilistic specifications, and can accommodate uncertainty originating from measurement limitations, expert assessment, forecasting variability, or incomplete observations. One of the important contributions to interval DEA was the development of interval efficiency assessment procedures that derive lower and upper efficiency bounds for DMUs whose production data cannot be represented by exact values [10]. This work established a useful methodological basis for incorporating bounded observations into DEA and demonstrated that efficiency analysis can remain computationally tractable even when inputs and outputs are imprecise.

The interval DEA literature subsequently expanded from efficiency assessment toward target setting and more general treatment of imprecise observations. Malekmohammadi et al. extended Wang-type interval representations to the determination of DEA targets, demonstrating the relevance of interval information not only for measuring performance but also for identifying attainable improvement levels [11]. More recently, generalized DEA models with imprecise data have provided broader formulations for accommodating uncertainty in production observations and have reinforced the importance of constructing frontier models capable of operating when exact input-output information is unavailable [12]. These developments are particularly pertinent to optimal system design because the design problem fundamentally depends on the benchmark technology. If benchmark observations themselves are uncertain, the input-output system derived from them should explicitly reflect that uncertainty rather than being constructed from arbitrarily selected point estimates.

Other strands of the DEA literature have approached uncertainty using robust, stochastic, and chance-constrained formulations. Robust DEA is especially useful when decision makers wish to protect efficiency evaluations or production decisions against unfavorable realizations of uncertain parameters. Data-driven robust DEA models, for example, have demonstrated how uncertainty sets informed by observed information can be incorporated into operational-efficiency assessment, thereby reducing dependence on unrealistically precise production data [13]. Robust DEA with variable budgeted uncertainty has further advanced this perspective by allowing the extent of protection against uncertainty to vary rather than imposing a single uniform level of conservatism [14]. Such approaches reveal an increasingly important principle in frontier analysis: uncertainty should be treated as an integral part of the production model rather than as a secondary issue addressed only after deterministic efficiency scores have been calculated.

Chance-constrained formulations provide another strategy when uncertainty can be characterized probabilistically. Research on optimal scale sizes and average-profit efficiency under uncertainty has shown how chance-constrained DEA can combine economic efficiency concepts with uncertain production conditions, thereby providing probabilistically feasible benchmarks and economically relevant scale recommendations [15]. Similarly, uncertain two-stage network DEA models based on chance-constrained programming illustrate how uncertainty can be incorporated when production consists of interconnected internal stages rather than a single black-box transformation process [16]. Collectively, these studies indicate that uncertainty is not confined to a particular DEA formulation; rather, it affects technical efficiency, economic performance, network production processes, system design, and resource-allocation decisions.

The importance of accommodating uncertainty becomes even greater in dynamic and network production environments. Contemporary organizations frequently consist of multiple linked activities whose outputs in one stage become intermediate products or resources for subsequent stages, while decisions in one period influence performance in later periods. Integrated dynamic interval DEA has demonstrated that interval information can be incorporated into multi-period frameworks even in the presence of challenging features such as integer and negative observations [17]. Dynamic network approaches have also increasingly integrated economic considerations into multi-stage and intertemporal efficiency evaluation. A directional-distance-function approach to cost-effectiveness in dynamic network DEA, for example, demonstrates how cost-related performance can be evaluated while acknowledging the internal and temporal structure of production systems [18]. More recently, value-based dynamic network DEA has integrated cost and revenue efficiency with target setting, further illustrating the movement of the field toward models that simultaneously evaluate performance and generate economically actionable production targets [19]. These developments underscore the relevance of integrating uncertainty, economic optimization, and target-oriented DEA rather than studying these issues as independent methodological problems.

Nevertheless, an important gap remains between the literature on optimal system design and the literature on uncertain or interval DEA. Existing profit-maximizing OSD models provide a powerful mechanism for constructing economically desirable production systems and investigating budget requirements, but their usefulness may be constrained when the reference input-output data are not precisely known. Conversely, interval and uncertainty-based DEA approaches have developed sophisticated procedures for handling imprecise observations, yet much of this literature remains oriented toward evaluating existing DMUs, estimating efficiency ranges, or setting conventional improvement targets. The direct integration of interval-valued benchmark data into a profit-maximizing OSD model therefore represents a meaningful methodological extension. Such integration permits the production system itself to be designed under bounded information rather than treating uncertainty merely as a complication surrounding an otherwise deterministic design.

This integration also creates a natural basis for revisiting the concept of optimal budget under uncertain production information. When input and output benchmarks fluctuate within intervals, the amount of financial resources required to support an economically optimal system may itself depend on how the uncertain production possibilities are represented. A useful model should consequently determine not only an optimal input-output configuration but also the minimum economically sufficient budget associated with maximum attainable profit. Furthermore, it should distinguish an increase in budget that expands economically productive opportunities from an increase that simply creates redundant financial capacity. The latter condition can be interpreted as budget congestion: a situation in which additional available resources fail to improve profit and, when full expenditure is imposed, may compel the system toward an economically inferior production configuration. This interpretation extends the logic of congestion from excessive physical input use toward excessive financial-resource allocation and provides an explicit managerial criterion for determining when additional expenditure is no longer economically justified.

Accordingly, a unified interval OSD framework can provide several methodological and practical benefits. First, it preserves the benchmarking logic of DEA while allowing reference production information to be expressed through lower and upper bounds. Second, it incorporates input costs, output prices, and budget constraints so that the resulting design is evaluated according to economic rather than purely technical criteria. Third, because the underlying structure can remain within linear programming, the model retains the computational tractability that has historically contributed to the practical applicability of DEA. Fourth, parametric analysis of the budget can be used to identify the level at which the maximum attainable profit is achieved and to determine whether further financial allocations provide additional economic value. Finally, the framework can supply managers with an interpretable distinction between an available budget and an optimal budget, thereby discouraging the assumption that complete expenditure of allocated resources is necessarily synonymous with efficient management.

Such a framework is potentially relevant across numerous sectors in which organizations must make resource-allocation decisions using imperfect production information. Banks may face uncertain demand and heterogeneous branch-level production conditions; healthcare organizations may need to allocate financial and human resources before service demand is precisely known; educational institutions may operate with uncertain enrollment, staffing requirements, and output measures; manufacturing organizations may confront variability in input requirements and production yields; and public agencies may be required to plan expenditures using forecasts and bounded estimates rather than exact future observations. In each of these contexts, an optimal design based on deterministic point estimates can provide misleading precision, whereas an interval framework more explicitly recognizes the informational conditions under which managerial planning occurs.

The literature therefore demonstrates substantial progress in economic DEA, optimal system design, interval efficiency analysis, robust and chance-constrained uncertainty modeling, dynamic and network structures, and economically oriented target setting. Yet the combination of these developments points to the need for a parsimonious framework capable of designing a profit-maximizing production system from interval-valued benchmark data while simultaneously determining an economically optimal budget and diagnosing excessive financial allocation. Addressing this need can extend the role of DEA from efficiency assessment toward uncertainty-aware economic system planning and provide a bridge between frontier-based benchmarking and practical budget-allocation decisions.

Therefore, the aim of this study is to develop an interval-data DEA framework for profit-maximizing optimal system design that determines optimal input-output targets and the optimal budget while identifying budget congestion under uncertainty.

2. Preliminaries

This section introduces the methodological foundations required for the proposed interval OSD model. First, the interval DEA approach is reviewed because it provides a convenient representation of imprecise production data. Second, the classical profit-maximizing OSD DEA model is presented as the deterministic benchmark that is extended in this paper.

Wang′s method

Assume that a set of DMUs consisting of $DMU_j; \quad j=1,\dots,n,$ with input–output vectors $(x_j, y_j); \quad j=1,\dots,n,$ in which $x_j=(x_{1j},x_{2j},\dots,x_{mj})^T$ and $y_j=(y_{1j},y_{2j},\dots,y_{sj})^T$. Define $X=(x_1,x_2,\dots,x_m)^T$ and $Y=(y_1,y_2,\dots,y_k)^T$ as $m\times n \;\; and \;\; s\times n$ matrices of inputs and outputs, respectively. Without the loss of generality, we assume that the input and output data $x_{ij} \;\; and \;\; y_{rj} \quad i=1,\dots,m \quad r=1,\dots,s \quad j=1,\dots,n,$ cannot be exactly obtained due to the existence of uncertainty. They are only known to lie within the upper and lower bounds represented by the ranges i.e. $x_{ij}\in\left[x_{ij}^L, x_{ij}^U\right]$ and $y_{rj}\in\left[y_{rj}^L, y_{rj}^U\right]$, where $x_{ij}^L\,(y_{rj}^L)$ and $x_{ij}^U\,(y_{rj}^U)$ are the lower and upper bounds of the intervals, and $x_{ij}^L>0$, $y_{ij}^L>0$. In this case, the upper bound and lower bound are obtained by two equivalent linear programming for the efficiency scores of the units as follows:

$$
\begin{aligned}
& Min\theta_{j_o}^L \qquad (1)\\
& s.t. \sum_{j=1}^{n}\lambda_j x_{ij}^L \le \theta_{j_o}^L x_{ij_o}^U, i=1,\dots,m,\\
& \sum_{j=1}^{n}\lambda_j y_{rj}^U \ge y_{rj_o}^L, r=1,\dots,s,\\
& \sum_{j=1}^{n}\lambda_j = 1\,,\\
& \lambda_j \ge 0, j=1,\dots,n.
\end{aligned}
$$

And

$$
\begin{aligned}
& Min\theta_{j_o}^{U} && (2)\\
& s.t.\sum_{j=1}^{n}\lambda_j x_{ij}^{L} \le \theta_{j_o}^{U} x_{ij_o}^{L}, i = 1,\dots,m, \\
& \sum_{j=1}^{n}\lambda_j y_{rj}^{U} \ge y_{rj_o}^{U}, r = 1,\dots,s, \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \ge 0, j = 1,\dots,n.
\end{aligned}
$$

Together with Models (1) and (2), $\theta_{j_o}^{L} and \theta_{j_o}^{U}$ constitutes an interval that covers all possible efficiency scores for the evaluated unit $DMUj_o$ and we define it as $\theta_{j_o}^{I} = \left[\theta_{j_o}^{L}, \theta_{j_o}^{U}\right]$.

Wang's method provides interval efficiency scores $\left[\theta_{j_o}^{L}, \theta_{j_o}^{U}\right]$ for each DMU under evaluation. However, the present study does not directly use these efficiency scores. Instead, we adopt Wang's representation of interval data — i.e., the use of lower and upper bounds for inputs and outputs — as the foundation for our OSD model. Specifically, Wang's method demonstrates how interval DEA models can be formulated using the bounds $x_{ij}^{L}, x_{ij}^{U}, y_{rj}^{L}, y_{rj}^{U}$. We extend this representation to the OSD context, where the goal is not to evaluate an existing DMU but to design an optimal production system. Thus, while Wang's efficiency intervals are not explicitly incorporated into our optimization model, the interval-data structure proposed by Wang et al. (2005) provides the methodological basis for handling uncertainty in the reference DMUs' production data.

2.1. Optimal system design DEA models

At this subsection, we provide an overview of the OSD DEA models proposed by Wei and Chang (2011b). suppose that the target DMU's price data are available, and let $C = (c_1, c_2, ..., c_m)^T > 0$ be an $m \times 1$ vector of input costs, and $P = \left(p_1, \dots, p_s\right)^T > 0$ be an $s \times 1$ vector of output prices. It should be noted that the reference DMUs' price data are unknown to the target DMU. Based upon the DMU's total available budget B, Wei and Chang (2011b) developed the following corresponding optimal Profit-maximization system design DEA model (a linear programming model):

$$
\begin{aligned}
& Max\sum_{r=1}^{s} p_r y_r - \sum_{i=1}^{m} c_i x_i \\
& (L_{OSD})\text{s.t.}\sum_{j=1}^{n}\hat{x}_{ij}\lambda_j \le x_i, i = 1,\dots,m, && (a)\\
& \sum_{j=1}^{n}\hat{y}_{rj}\lambda_j \ge y_r, r = 1,\dots,s, && (b)\\
& \sum_{i=1}^{m} c_i x_i \le B, \\
& \sum_{j=1}^{n}\lambda_j \le 1, \\
& x_i \ge 0, y_r \ge 0, i = 1,\dots,m, r = 1,\dots,s,
\end{aligned}
$$

$$\lambda_j \geq 0, j = 1, \ldots, n.$$

where $\hat{x}_j = (\hat{x}_{1j}, \hat{x}_{2j}, \hat{x}_{mj})^T > 0$, the input vector for the $j\,th$ DMU; and $\hat{y}_j = (\hat{y}_{1j}, \hat{y}_{2j}, \ldots, \hat{y}_{Sj})^T > 0$, the output vector for the $j\,th$ DMU, for $j = 1,\ldots,n.$

LOSD included three types of decision variables, named, input vector x, output vector y and weight vector λ. Moreover, it is should be noted that the production possibility sets act as the feasible regions in the OSD DEA models which the n observed DMUs' input and output information is used to benchmark DMU–(x,y). It could be concluded that the larger number of observed DMUs, gives the more effective the benchmarking ability.

From inequalities (a) and (b), and the condition that C > 0 and P > 0, it is easy to check that the optimal solution of (LOSD), that maximizes $\sum_{r=1}^{s} p_r y_r - \sum_{i=1}^{m} c_i x_i$ must satisfy

$$x_i = \sum_{j=1}^{n} \lambda_j \hat{x}_{ij}, \quad i = 1,\ldots,m \quad , \qquad y_r = \sum_{j=1}^{n} \lambda_j \hat{y}_{rj} \quad , \quad r = 1,\ldots,s.$$

Therefore, by replacing x_i and y_r with $\sum_{j=1}^{n} \lambda_j \hat{x}_{ij}$ *and* $\sum_{j=1}^{n} \lambda_j \hat{y}_{rj}$, respectively, In (LOSD), we reach the following linear programming model, that only consists of two constraints and n non-negative variables, i.e., $\lambda_1, \lambda_2, \ldots, \lambda_n$.:

$$
\begin{aligned}
& Max \sum_{j=1}^{n} \lambda_j \sum_{r=1}^{s} p_r \hat{y}_{rj} - \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{m} c_i \hat{x}_{ij} \\
& \text{s.t.} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3) \\
& \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{m} c_i \hat{x}_{ij} \leq B. \\
& \sum_{j=1}^{n} \lambda_j \leq 1 \\
& \lambda_j \geq 0, j = 1, \ldots, n
\end{aligned}
$$

Definition 1. Denote the optimal solution to (3) as λ^* the optimal solution (3), and as x^* *and* y^* the corresponding optimal values of x and y, respectively; note that $\sum_{r=1}^{n} \hat{x}_j \lambda_j^* = x^*$ and $\sum_{r=1}^{n} \hat{y}_j \lambda_j^* = y^*$. (x^*, y^*) is referred to as the optimal design (based on benchmarking n DMUs).

Definition 2. The system associated with optimal design (x^*, y^*) given total available budget B is referred to as the optimal system because (x^*, y^*) is the optimal input–output pair that a system can have under these circumstances.

3. Optimal System Design for DMUs with Interval data

To address uncertainty in production data, we extend the optimal profit-maximization Optimal System Design (OSD) DEA model to the case where the inputs and outputs of the reference DMUs are represented by interval values. Following the interval DEA framework proposed by Wang et al. (2005); Malekmohammadi et al., 2010; Wu et al., 2024, each input and output is assumed to be known only within a lower and an upper bound rather than as an exact value. This interval representation provides a practical way to model uncertainty arising from measurement errors, incomplete observations, forecasting uncertainty, and managerial judgment.

Let $j = 1, \dots, n$ denote the set of DMUs, $i = 1, \dots, m$ the input indices, and $r = 1, \dots, s$ the output indices. Let x_{ij} denote the amount of input i consumed by DMU_j, and y_{rj} denote the amount of output r produced by DMU_j. Under interval uncertainty, the production data are represented as

$$x_{ij} \in \left[x_{ij}^L, x_{ij}^U\right], \quad y_{rj} \in \left[y_{rj}^L, y_{rj}^U\right]$$

where, x_{ij}^L and y_{rj}^L denote the lower and upper bounds of input i, respectively, and y_{rj}^L and y_{rj}^U denote the corresponding lower and upper bounds of output r. It is assumed that $x_{ij}^L \geq 0$ and $y_{rj}^L \geq 0$ for all $i = 1, \dots, m$, $r = 1, \dots, s$, and $j = 1, \dots, n$.

Assume that the target DMU has complete information regarding its own economic parameters. Let

$$C = (c_1, c_2, \dots, c_m)^T > 0,$$

be the m×1 vector of input costs, and

$$P = (p_1, p_2, \dots, p_s)^T > 0,$$

be the s×1 vector of output prices. As in the classical OSD DEA framework of Wei and Chang (2011b), the price information of the reference DMUs is assumed to be unavailable. Given the total available budget B, the following linear programming model is proposed to determine the optimal profit-maximizing system design under interval data.

In Model (L_{OSD}^1) below, we adopt an optimistic bound-selection strategy: inputs are represented by their lower bounds x_i^L and outputs by their upper bounds y_r^U. This choice is deliberate and reflects the planning context of optimal system design. Unlike traditional DEA evaluation models — which assess the performance of existing DMUs using consistent bounds for the same unit — our OSD framework is concerned with designing a new production system. In this design context, the target DMU is not restricted to any single observed realization of the uncertain data; rather, it can plan for a favorable combination of input and output levels as long as each lies within its respective interval.

Consequently, the optimistic bound selection corresponds to a best-case planning scenario, consistent with the maximax decision criterion, and is common in interval DEA studies that focus on identifying upper-bound benchmark targets (Wang et al., 2005; Malekmohammadi et al., 2010; Wu et al., 2024).

It is important to emphasize that the proposed model does not claim to evaluate the actual performance of the target DMU under any specific realization of the data. Instead, it provides a target design that the DMU can aim to achieve if favorable conditions materialize. For a conservative (worst-case) design, one could instead use x_i^L for inputs and y_r^U for outputs; such an extension is straightforward and would yield a lower bound on attainable profit. The choice between optimistic and conservative designs depends on the risk attitude of the decision maker and the specific planning context.

(L_{OSD}^1)

$$
\begin{aligned}
& Max \sum_{r=1}^{s} p_r y_r^U - \sum_{i=1}^{m} c_i x_i^L \\
& \text{s.t.} \sum_{j=1}^{n} \hat{x}_{ij}^L \lambda_j \le x_i^L, \qquad i = 1,2,\ldots,m, \qquad (I) \\
& \sum_{j=1}^{n} \hat{y}_{rj}^U \lambda_j \ge y_r^U, \qquad r = 1,\ldots,s, \qquad \text{(II)} \\
& \sum_{i=1}^{m} c_i x_i^L \le B, \\
& \sum_{j=1}^{n} \lambda_j \le 1\,, \\
& x_i^L \ge 0, \qquad i = 1,\ldots,m, \\
& \qquad y_r^U \ge 0, \qquad r = 1,\ldots,s \\
& \qquad \lambda_j \ge 0, \qquad j = 1,\ldots,n.
\end{aligned}
$$

Model (L_{OSD}^1) contains three groups of decision variables: the input vector x, the output vector y, and the intensity vector λ. The model constructs a virtual production unit as a convex combination of the observed DMUs and determines the input-output combination that maximizes the target DMU's profit while satisfying the available budget constraint. Similar to the classical OSD DEA framework, the production possibility set is generated from the observed interval-valued production data of the reference DMUs. Consequently, increasing the number of observed DMUs enriches the production possibility set and enhances the benchmarking capability of the proposed model.

Since the input costs and output prices are strictly positive (C>0 and P>0), any unnecessary increase in inputs or decrease in outputs would reduce the objective function. Therefore, at the optimal solution, the production constraints become binding. Hence, the optimal solution of Model (L_{OSD}^1) satisfies

$$x_i^L = \sum_{j=1}^{n} \lambda_j \hat{x}_{ij}^L\,, \qquad i = 1,,\ldots,m,$$

and

$$y_r^U = \sum_{j=1}^{n} \lambda_j \hat{y}_{rj}^U\,, \qquad r = 1,\ldots,s.$$

To verify that the optimal solution satisfies these equalities, suppose, to the contrary, that there exists an optimal solution with

$$x_i^L > \sum_{j=1}^{n} \lambda_j \hat{x}_{ij}^L$$

for some input i. Since $c_i > 0$, reducing x_i^L to $\sum_{j=1}^{n} \lambda_j \hat{x}_{ij}^L$ would strictly decrease the term $\sum_{i=1}^{m} c_i\, x_i^L$ in the objective function without violating any constraint, thereby increasing the objective value. This contradicts the optimality of the solution. Therefore, $x_i^L = \sum_{j=1}^{n} \lambda_j \hat{x}_{ij}^L$ must hold at optimality.

Similarly, suppose there exists an optimal solution with

$$y_r^U < \sum_{j=1}^{n} \lambda_j \hat{y}_{rj}^U$$

for some output r. Since $p_r > 0$, increasing y_r^U to $\sum_{j=1}^{n} \lambda_j \hat{y}_{rj}^U$ would strictly increase the term $\sum_{r=1}^{s} p_r\, y_r^U$ in the objective function without violating any constraint, thereby increasing the objective value. This contradicts the optimality of the solution. Therefore, $y_r^U = \sum_{j=1}^{n} \lambda_j \hat{y}_{rj}^U$ must hold at optimality.

Hence, at the optimal solution, both the input and output constraints are binding, and no slack variables are present. This justifies the substitution of x_i^L and y_r^U with their corresponding convex combinations of the observed bounds.

Substituting these equalities into the objective function and the budget constraint eliminates the explicit input and output decision variables, resulting in an equivalent optimization model involving only the intensity variables.

Accordingly, Model (L_{OSD}^1) can be reformulated as

$$
\begin{aligned}
& Max \sum_{j=1}^{n} p_r \left(\sum_{j=1}^{n} \lambda_j\, \hat{y}_{rj}^U \right) - \sum_{i=1}^{m} c_i \left(\sum_{j=1}^{n} \lambda_j\, \hat{x}_{ij}^L \right) \\
& \text{s.t.} \\
& \sum_{i=1}^{m} c_i \left(\sum_{j=1}^{n} \lambda_j\, \hat{x}_{ij}^L \right) \le B, \\
& \sum_{j=1}^{n} \lambda_j \le 1\,, \\
& \lambda_j \ge 0, \qquad j = 1, \ldots, n.
\end{aligned}
$$

which is denoted by (L_{OSD}^2).

By rearranging the objective function of Model (L_{OSD}^2), the optimization problem can be expressed in a more compact form. Specifically, by grouping the terms associated with each reference DMU, Model (L_{OSD}^2) is equivalently rewritten as

$$
\begin{aligned}
& Max \sum_{j=1}^{n} \lambda_j \left(\sum_{r=1}^{s} p_r\, \hat{y}_{rj}^U - \sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right) \\
& \text{s.t.} \\
& \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} c_i\, \hat{x}_{ij}^L \right) \le B, \\
& \sum_{j=1}^{n} \lambda_j \le 1\,,
\end{aligned}
$$

$\lambda_j \ge 0,\ j = 1, \ldots, n.$

Compared with Model (L_{OSD}^1), the compact formulation (L_{OSD}^3) contains only one class of decision variables, namely the intensity variables λ_j. Consequently, the computational burden is substantially reduced without changing the optimal objective value or the corresponding optimal production plan. Therefore, Model (L_{OSD}^3) is computationally preferable for determining the optimal system design.

After solving Model (L_{OSD}^3), the optimal intensity vector

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$$

is obtained. The corresponding projected production point is then computed as the convex combination of the observed DMUs using the optimal intensity variables.

Accordingly, the projected input and output vectors are determined by

$$\hat{x}_i = \sum_{j=1}^{n} \lambda_j^* \hat{x}_{ij}^L, \qquad i = 1,2,\dots,m,$$

and

$$\hat{y}_r = \sum_{j=1}^{n} \lambda_j^* \hat{y}_{rj}^U, \qquad r = 1,2,\dots,s.$$

The projected point

$$(\hat{x}, \hat{y})$$

represents the optimal profit-maximizing production system that satisfies the available budget constraint and lies on the efficient frontier generated by the observed DMUs. Therefore, it provides the benchmark production plan for the target DMU under the interval-data framework.

Definition 3. Let

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$$

denote the optimal solution of Model (L_{OSD}^3). The corresponding optimal input and output vectors are respectively given by

$$x_i^* = \sum_{j=1}^{n} \lambda_j^* \hat{x}_{ij}^L, \qquad i = 1,2,\dots,m,$$

and

$$y_r^* = \sum_{j=1}^{n} \lambda_j^* \hat{y}_{rj}^U, \qquad r = 1,2,\dots,s.$$

The corresponding budget consumed by the optimal system design is

$$B^* = \sum_{i=1}^{m} c_i x_i^* = \sum_{j=1}^{n} \lambda_j^* \left(\sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right).$$

The maximum attainable profit associated with the available budget B is defined as

$$f(B) = \sum_{r=1}^{s} p_r y_r^* - \sum_{i=1}^{m} c_i x_i^*,$$

or, equivalently,

$$f(B) = \sum_{j=1}^{n} \lambda_j^* \left(\sum_{r=1}^{s} p_r \hat{y}_{rj}^U - \sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right).$$

Accordingly, the pair

$$(x^*, y^*)$$

is referred to as the optimal system design, while

$$B - B^*$$

represents the remaining budget after implementing the optimal production plan.

The optimal input-output pair (x^*, y^*) defines the profit-maximizing production system attainable under the available budget B. Therefore, it provides the optimal benchmark for the target DMU within the proposed interval optimal system design framework.

Theorem 1. Let

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$$

be the optimal solution of Model (L_{OSD}^3), and let

$$x_i^* = \sum_{j=1}^{n} \lambda_j^* \hat{x}_{ij}^L, \qquad i = 1,2, \dots, m,$$

and

$$y_r^* = \sum_{j=1}^{n} \lambda_j^* \hat{y}_{rj}^U, \qquad r = 1,2, \dots, s.$$

be the corresponding optimal input and output vectors. Then, the optimal system design

$$(x^*, y^*)$$

lies on the efficient frontier of the production possibility set T_{OSD}. Equivalently, (x^*, y^*) is a Pareto-efficient solution of the following vector optimization problem:

$$(VP) \qquad min(x, -y)$$
$$s.t. \qquad (x, y) \in T_{OSD}.$$

Proof: Let

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$$

be an optimal solution of Model (L_{OSD}^3). By Definition 3, the corresponding optimal input-output vectors are

$$x_i^* = \sum_{j=1}^{n} \lambda_j^* \hat{x}_{ij}^L, \qquad i = 1,2, \dots, m,$$

and

$$y_r^* = \sum_{j=1}^{n} \lambda_j^* \hat{y}_{rj}^U, \qquad r = 1,2, \dots, s.$$

Hence, the pair

$$(x^*, y^*)$$

is feasible for the production possibility set T_{OSD}.

The objective function of Model (L_{OSD}^3) is

$$\max\left(\sum_{r=1}^{s} p_r y_r - \sum_{i=1}^{m} c_i x_i\right),$$

where the output price vector $P = (p_1, \dots, p_s)^T$ and the input cost vector $C = (c_1, \dots, c_m)^T$ satisfy

$$P > 0, \qquad C > 0.$$

Therefore, the objective function is strictly increasing with respect to each output component and strictly decreasing with respect to each input component. Consequently, any feasible solution that improves at least one output without increasing any input, or decreases at least one input without reducing any output, would yield a strictly larger objective value, contradicting the optimality of (x^*, y^*).

Hence, there exists no feasible point in T_{OSD} that dominates (x^*, y^*). Therefore,

$$(x^*, y^*)$$

is a Pareto-efficient solution of the vector optimization problem

$$min(x, -y)$$

subject to

$$(x, y) \in T_{OSD}.$$

According to the fundamental results of multi-objective optimization (Zeleny, 1974), every Pareto-efficient solution lies on the efficient frontier of the production possibility set. Consequently,

$$(x^*, y^*)$$

belongs to the efficient frontier of T_{OSD}.□

The OSD DEA Model (L_{OSD}^{3}) ccan be viewed as a parametric linear programming (PLP) problem in which the total available budget B appears as the right-hand-side parameter of the budget constraint. Therefore, the theory of parametric linear programming (Gass, 2010; Gass and Saaty, 1955; Wei and Chang, 2011b) can be employed to investigate the relationship between the optimal system design and the available budget, and to determine the target DMU's optimal budget. This issue is discussed in detail in the section 5.

Example

To illustrate the applicability of the proposed interval OSD DEA model, we consider a data set consisting of 10 decision-making units (DMUs), each characterized by three inputs and two outputs. Since the production data are subject to uncertainty, all inputs and outputs are represented by interval values. The interval input-output data of the ten DMUs are reported in Table 1.

The input cost vector and output price vector are specified as

$$C = (0.5,0.1,0.5)^{T},$$

and

$$P = (0.2,1)^{T},$$

respectively. These values are assumed to be known by the target DMU and remain fixed throughout the analysis. Following the assumptions of the proposed OSD DEA model, the economic information of the reference DMUs is not required.

Table1. Interval input-output data of the 10 DMUs.

DMU	x_1	x_2	x_3	y_1	y_2
1	[7,11]	[40,60]	[1,3]	[80,120]	[25,45]
2	[8,12]	[30,50]	[3,5]	[60,100]	[5,15]
3	[7,11]	[20,40]	[2,4]	[80,120]	[20,40]
4	[6,10]	[15,35]	[1,3]	[40,80]	[5,25]
5	[8,12]	[30,50]	[4,6]	[65,105]	[15,35]
6	[5,9]	[25,45]	[1,4]	[70,110]	[25,45]
7	[5,9]	[20,40]	[4,6]	[80,120]	[15,35]
8	[10,14]	[30,50]	[3,5]	[100,140]	[5,15]
9	[7,11]	[15,35]	[1,4]	[90,130]	[5,25]

10	[8,12]	[40,60]	[1,3]	[60,100]	[10,30]

Model (L^3_{OSD}) was solved for each DMU using the proposed optimization procedure. For every target DMU, the model determines the corresponding optimal input-output combination that maximizes profit while satisfying the available budget constraint. The resulting optimal system designs are summarized in Table 2.

Specifically, Columns 2–4 present the optimal levels of the three input variables, Columns 5 and 6 report the corresponding optimal output levels, and the final column presents the maximum attainable profit associated with each optimal system design.

Table 2. Optimal system designs and corresponding maximum profits.

DMU	x_1^*	x_2^*	x_3^*	y_1^*	y_2^*	*profit*
1	5.50	27.50	0.69	68.75	20.62	41.94
2	5.18	25.88	0.65	64.70	19.41	17.15
3	6.29	31.43	0.79	78.57	23.57	44.79
4	8.00	40.00	1.00	100.00	30.00	36.00
5	4.89	24.44	0.61	61.11	18.33	28.72
6	8.00	40.00	1.00	100.00	30.00	61.50
7	6.77	33.85	0.85	84.61	25.38	44.42
8	4.63	23.16	0.58	57.89	17.37	19.39
9	8.00	40.00	1.00	100.00	30.00	45.50
10	5.18	25.88	0.65	64.71	19.41	26.85

The computational results indicate that the proposed interval OSD DEA model successfully identifies an economically efficient production plan for each DMU. The optimal system designs differ considerably across the ten DMUs, reflecting differences in their production structures and interval-valued production data. Likewise, the corresponding maximum profits vary substantially, indicating that the economic potential of the DMUs is strongly influenced by both their production technology and the available budget.

For example, DMU 6 achieves the highest maximum profit among the ten DMUs, whereas DMU 2 produces the lowest profit. This observation suggests that the proposed model is capable of distinguishing the economic performance of DMUs while simultaneously accounting for uncertainty in the production data.

The results further demonstrate that the proposed interval OSD DEA model provides both an optimal production target and its associated economic performance. Consequently, the proposed approach serves not only as a benchmarking tool but also as a decision-support model for determining economically optimal production systems under interval uncertainty.

4. Optimal budget determination and budget congestion

Determining the optimal budget is one of the fundamental issues in optimal system design because the available budget directly influences the attainable production plan and the corresponding economic performance. This issue becomes even more important when the production system is characterized by interval-valued inputs and outputs, since uncertainty in production data affects both resource allocation and the resulting production targets. Consequently, identifying an appropriate budget under interval uncertainty is an essential component of the proposed interval OSD DEA framework.

In addition to determining the optimal budget, it is equally important to investigate whether increasing the available budget always improves the economic performance of the system. In practice, additional financial resources do not necessarily lead to proportional improvements in production efficiency or profitability. Beyond a certain level of investment, additional budget may generate little or no improvement in the optimal production plan. This phenomenon is commonly referred to as budget congestion.

To address these issues, this section is divided into two parts. First, a parametric linear programming approach is employed to determine the optimal budget associated with the proposed interval OSD model. Subsequently, the concept of budget congestion is investigated to identify situations in which additional budget fails to produce corresponding improvements in the optimal system design.

4.1. Optimal budget determination under uncertainty condition

The total available budget B plays a fundamental role in determining the optimal system design (x^*, y^*) obtained from Model (L^3_{OSD}). Since the budget constraint directly affects the feasible region of the optimization problem, it also influences both the optimal objective value and the corresponding optimal input-output combination. Therefore, understanding the relationship between the available budget and the optimal system performance is essential for effective resource allocation under interval uncertainty. Let Z denote the optimal objective value of Model (L^3_{OSD}). By treating the available budget B as a parameter, the optimal objective value can be expressed as a function of the budget, namely,

$$Z = f(B),$$

where f(B) represents the maximum attainable profit corresponding to the available budget B. The domain of the budget function is defined as

$$\Omega = \{ B \in \mathbb{R} \mid B \geq 0 \}.$$

Accordingly, Model (L^3_{OSD}) can be written as the following parametric linear programming problem:

$$(L^3_{OSD}) \quad Max \sum_{j=1}^{n} \lambda_j \left(\sum_{r=1}^{s} p_r \hat{y}^U_{rj} - \sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right)$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right) \leq B,$$

$$\sum_{j=1}^{n} \lambda_j \leq 1$$
$$\lambda_j \geq 0, \qquad j = 1, \ldots, n.$$

As shown by Wei and Chang (2011a), the optimal objective value of the OSD DEA model is a monotonically non-decreasing and concave function of the available budget. Consequently, increasing the available budget cannot decrease the maximum attainable profit; however, additional budget does not necessarily result in proportional improvements in the optimal objective value. Beyond a certain budget level, further increases in available resources may have little or no impact on the optimal profit.

From a managerial perspective, it is therefore important to determine whether allocating additional budget is economically beneficial. In particular, decision makers need to identify the minimum budget required to achieve the maximum attainable profit and to determine whether the available budget is being utilized efficiently. These considerations naturally lead to the concepts of optimal budget and budget congestion, which are formally introduced in the following definitions and theorems.

Definition 4. Let f(B) denote the optimal objective value of Model (L^3_{OSD}) corresponding to the available budget B. The optimal budget is defined as

$$B^{**} = \min\left\{B \in \Omega \,\middle|\, f(B) = \max_{B \in \Omega} f(B)\right\},$$

where

$$\Omega = \{\mathrm{B} \in \mathbb{R} | B \geq 0\}$$

is the feasible set of available budgets.

In other words, B^{**} is the smallest available budget that attains the maximum profit of the proposed interval OSD model. Therefore, any budget greater than B^{**} does not improve the optimal objective value and is unnecessary from the viewpoint of profit maximization.

Theorem 2. Let

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$$

be the optimal solution of Model (L^3_{OSD}), and suppose that

$$\sum_{j=1}^{n} \lambda_j^* \left(\sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right) < B.$$

Define

$$B^{**} = \sum_{j=1}^{n} \lambda_j^* \left(\sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right).$$

Then, for every available budget

$$\bar{B} \geq B^{**},$$

the optimal objective value remains unchanged; that is,

$$f(\bar{B}) = f(B^{**}) = \sum_{j=1}^{n} \lambda_j^* \left(\sum_{r=1}^{s} p_r \hat{y}_{rj}^U - \sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right).$$

Consequently, B^{**} is the optimal budget of the target DMU, since any additional budget beyond B^{**} does not increase the maximum attainable profit.

To determine the optimal budget, it is useful to consider the OSD DEA model without an explicit budget constraint. By removing the budget restriction from Model (L^3_{OSD}), the following optimization problem is obtained:

$$(L^4_{OSD}) \qquad \max f(B) = \sum_{j=1}^{n} \lambda_j \left(\sum_{r=1}^{s} p_r \hat{y}^U_{rj} - \sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right)$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \leq 1$$

$$\lambda_j \geq 0, \qquad j = 1, \ldots, n.$$

Model (L^4_{OSD}) represents the unconstrained optimal system design problem, in which the available budget does not restrict the feasible production set. Consequently, the model determines the maximum attainable profit independently of any budget limitation. According to Theorem 2, the optimal solution of Model (L^4_{OSD}) provides the optimal intensity vector λ^*, from which the corresponding optimal budget B^{**} and the associated optimal input-output vectors can be obtained directly. Specifically, the optimal budget is computed as

$$B^{**} = \sum_{j=1}^{n} \lambda^*_j \left(\sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right),$$

while the corresponding optimal system design is determined according to Definition 3.

Therefore, solving Model (L^4_{OSD}) enables the decision maker to identify both the minimum budget required to achieve the maximum attainable profit and the corresponding optimal production plan under interval uncertainty. This result follows directly from Theorem 2 and establishes the basis for the subsequent analysis of budget congestion.

Let f (B*) denote the optimal objective value of Model (L^3_{OSD}). Since Model (L^3_{OSD}) is a linear programming problem, multiple optimal solutions may exist. Consequently, different optimal solutions may consume different amounts of budget while yielding the same maximum attainable profit.

To characterize the range of budget values associated with the optimal profit, two auxiliary optimization models are proposed. The first model determines the minimum budget required to attain the optimal objective value, whereas the second model determines the maximum budget corresponding to the same optimal profit.

The minimum feasible budget is obtained by solving the following optimization problem:

$$Min B_L \tag{7}$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \left(\sum_{r=1}^{s} p_r \hat{y}^U_{rj} - \sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right) = f(B^*),$$

$$\sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right) = B_L,$$

$$\sum_{j=1}^{n} \lambda_j \leq 1\,,$$

$$\lambda_j \geq 0, \qquad j = 1, \ldots, n.$$

Similarly, the maximum feasible budget associated with the same optimal objective value is obtained from

$$Max\ B_U \tag{8}$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \left(\sum_{r=1}^{s} p_r \hat{y}^U_{rj} - \sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right) = f(B^*),$$

$$\sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} c_i \hat{x}_{ij}^{L} \right) = B_U,$$
$$\sum_{j=1}^{n} \lambda_j \leq 1\,,$$
$$\lambda_j \geq 0, \qquad j = 1, \ldots, n.$$

Let B_L^* and B_U^* denote the optimal solutions of Models (7) and (8), respectively. These two values define the interval of budget levels that achieve the same maximum profit. Therefore, when multiple optimal solutions exist, any available budget satisfying

$$B_L^* \leq B \leq B_U^*$$

can support an optimal production plan without reducing the optimal objective value.

If the available budget is assumed to be fully utilized, the proposed interval OSD DEA model can be reformulated as

$$\begin{aligned} & Max\ f(B) = \sum_{j=1}^{n} \lambda_j \left(\sum_{r=1}^{s} p_r \hat{y}_{rj}^{U} - \sum_{i=1}^{m} c_i \hat{x}_{ij}^{L} \right) \\ & \text{s.t.} \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} c_i \hat{x}_{ij}^{L} \right) = B, \\ & \sum_{j=1}^{n} \lambda_j \leq 1 \\ & \lambda_j \geq 0, \qquad j = 1, \ldots, n. \end{aligned} \tag{9}$$

Model (9) represents the interval OSD DEA formulation under the assumption that the entire available budget is allocated to the production system. This formulation provides the basis for the subsequent analysis of budget congestion under interval uncertainty.

Theorem 3. Let B_L^* and B_U^* denote the optimal solutions of Models (7) and (8), respectively. Then, the optimal objective value f(B) of Model (L_{OSD}^3) satisfies the following properties:

If

$$B < B_L^*,$$

then f(B) is a monotonically increasing function of the available budget B.

If

$$B_L^* \leq B \leq B_U^*,$$

then

$$\text{f(B)} = \text{f}(B^*),$$

that is, the maximum attainable profit remains constant throughout the interval $[B_L^*, B_U^*]$.

If

$$B > B_U^*,$$

then

$$\text{f(B)} = \text{f}(B^*),$$

and therefore, additional budget does not improve the optimal objective value. Consequently, any increase in the available budget beyond B_U^* is economically ineffective with respect to profit maximization.

The results presented in Theorems 2 and 3 are consistent with the findings of Fang (2014), while extending the analysis to the interval-data Optimal System Design (OSD) DEA framework. Based on these results, the following corollary is obtained.

Corollary: For Model (L^3_{OSD}), the optimal budget is B^{**}, and the corresponding maximum attainable profit is $f\ (B^{**})$. Furthermore, a target DMU is said to experience budget congestion whenever

$$B > B^*_U,$$

where B denotes the available budget and B^*_U, is the upper bound of the optimal budget interval obtained from Model (8). Under this condition, increasing the available budget does not improve the maximum attainable profit; that is,

$$f(B) = f(B^{**}), \qquad \forall\ B > B^*_U.$$

Therefore, the additional budget allocated beyond B^*_U does not contribute to improving the economic performance of the target DMU and is regarded as budget congestion within the proposed interval OSD DEA framework.

4.2. *Budget congestion*

In the proposed interval OSD DEA framework, the available budget is treated as a limited economic resource that determines the feasible production system. The previous section showed that increasing the available budget does not necessarily improve the maximum attainable profit. Consequently, an important managerial question is whether allocating and fully consuming the available budget is always economically beneficial.

According to Theorem 2, when the budget consumed by the optimal system satisfies

$$\sum_{i=1}^{m} c_i x_i^L \leq B,$$

additional budget does not change the optimal production plan or increase the maximum attainable profit. Therefore, forcing the decision maker to utilize the entire available budget may lead to inefficient resource allocation without improving economic performance. This phenomenon is referred to as budget congestion in the proposed interval OSD DEA framework.

Unlike the classical notion of congestion in DEA, which is associated with excessive use of production inputs, budget congestion describes the situation in which additional financial resources fail to generate any improvement in the optimal profit. Consequently, allocating a budget beyond the economically efficient level results in unnecessary resource consumption without increasing the system's performance.

To investigate this phenomenon, suppose that the available budget must be completely utilized. Replacing the budget inequality

$$\sum_{i=1}^{m} c_i x_i^L \leq B$$

with the equality

$$\sum_{i=1}^{m} c_i x_i^L = B,$$

the proposed interval OSD DEA model becomes

$$(\hat{L}^1_{OSD}) \qquad Max f(B) = \sum_{r=1}^{s} p_r y_r^U - \sum_{i=1}^{m} c_i x_i^L$$

$$\text{s.t.} \qquad \sum_{j=1}^{n} \lambda_j \hat{x}_{ij}^L \leq x_i, \qquad i = 1, \dots, m,$$

$$\sum_{j=1}^{n} \lambda_j \hat{y}_{rj}^U \geq y_r, \qquad r = 1, \ldots, s,$$
$$\sum_{i=1}^{m} c_i x_i^L = B,$$
$$\sum_{j=1}^{n} \lambda_j \leq 1\,,$$
$$x_i \geq 0, \qquad i = 1, \ldots, m,$$
$$y_r \geq 0, \qquad r = 1, \ldots, s,$$
$$\lambda_j \geq 0, \qquad j = 1, \ldots, n.$$

Eliminating the explicit input and output variables in the same manner as described for Model (L_{OSD}^1), an equivalent formulation is obtained as

$$\left(\hat{L}_{OSD}^3\right) \qquad Max \hat{f}(B) = \sum_{j=1}^{n} \lambda_j \left(\sum_{r=1}^{s} p_r \hat{y}_{rj}^U - \sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right)$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right) = B,$$
$$\sum_{j=1}^{n} \lambda_j \leq 1$$
$$\lambda_j \geq 0, \qquad j = 1, \ldots, n.$$

Assume that Model $\left(\hat{L}_{OSD}^3\right)$ is feasible for the given budget B, and let

$$\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$$

be its optimal solution. Suppose further that

$$B^* = \sum_{j=1}^{n} \bar{\lambda}_j \left(\sum_{i=1}^{m} c_i \hat{x}_{ij}^L \right) < B.$$

In this case, the optimal objective value obtained under the compulsory budget-consumption assumption satisfies

$$\hat{f}(B) \leq f(B).$$

Hence, requiring the entire available budget to be spent results in an economically inferior solution compared with the original interval OSD model. Therefore, whenever

$$B > B^*,$$

the target DMU experiences budget congestion, since additional budget does not improve the maximum attainable profit and may reduce the economic efficiency of the production system.

The inequality $\hat{f}(B) \leq f(B)$ is a mathematical consequence of the fact that the equality-constrained feasible set is a subset of the inequality-constrained feasible set. However, the economic interpretation of budget congestion goes beyond this mathematical observation. In the proposed framework, budget congestion is said to occur when the marginal profit with respect to the available budget becomes zero or negative; that is, when $\frac{df(B)}{dB} \leq 0$.

This occurs precisely when the available budget exceeds the optimal budget B^{**} (or B_U^* in the case of multiple optimal solutions). At this point, additional financial resources provide no further improvement in profit, indicating that the DMU has reached its economically efficient scale of operation. This phenomenon is analogous to the classical concept of congestion in production economics, where additional inputs fail to increase output, but is adapted here to the context of budget allocation and profit maximization.

Theorem 4. Let

$$\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_n)$$

be the optimal solution of Model (L^3_{OSD}), and assume that

$$B^* = \sum_{j=1}^{n} \bar{\lambda}_j \left(\sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right) \leq B.$$

Then,

$$\hat{f}(B) \leq f(B) = f(B^*).$$

Proof: Let

$$S_1 = \left\{ x \,\middle|\, \sum_{i=1}^{m} c_i x_i^L = B \right\}$$

and

$$S_2 = \left\{ x \,\middle|\, \sum_{i=1}^{m} c_i x_i^L \leq B \right\}$$

denote the feasible regions of Models (L^1_{OSD}) and $(\hat{L}^3_{OSD})$, respectively. Since every feasible solution satisfying

$$\sum_{i=1}^{m} c_i x_i^L = B$$

also satisfies

$$\sum_{i=1}^{m} c_i x_i^L \leq B,$$

it follows immediately that

$$S_1 \subseteq S_2.$$

Therefore, the feasible region of the equality-constrained model is a subset of the feasible region of the inequality-constrained model. Since both models maximize the same objective function, restricting the feasible region cannot improve the optimal objective value. Consequently,

$$\hat{f}(B) \leq f(B).$$

Furthermore, by Theorem 2, whenever

$$B^* \leq B,$$

the optimal objective value of the original interval OSD DEA model remains unchanged; that is,

$$f(B) = f(B^*).$$

Combining the above two results yields

$$\hat{f}(B) \leq f(B) = f(B^*)$$

which completes the proof. □

The equality constraint

$$\sum_{j=1}^{n} \bar{\lambda}_j \left(\sum_{i=1}^{m} c_i \hat{x}^L_{ij} \right) = B$$

in Model $(\hat{L}^3_{OSD})$ requires the target DMU to utilize the entire available budget, regardless of whether doing so improves its economic performance. Consequently, the imposed equality may force the DMU to operate beyond its economically efficient budget level, thereby giving rise to budget congestion.

Within the proposed interval OSD DEA framework, budget congestion refers to the situation in which an increase in the available budget beyond the optimal budget does not increase the maximum attainable profit and may even reduce the economic efficiency of the production system. In such cases, the marginal benefit obtained from additional expenditure is insufficient to compensate for the corresponding increase in production cost, resulting in economically inefficient resource allocation.

The previous analysis established that the original model allows the DMU to determine its optimal budget endogenously through the inequality budget constraint. In contrast, Model $(\hat{L}_{OSD}^{3})$ forces the entire available budget to be consumed. Consequently, whenever the available budget exceeds the economically optimal level, the equality-constrained model may produce a lower objective value than the original model. This phenomenon provides the theoretical basis for identifying budget congestion within the proposed interval OSD DEA framework.

To illustrate the occurrence and economic implications of budget congestion, a numerical example is presented in the following subsection.

5. Illustrative Example

To illustrate the proposed interval OSD DEA approach, the same data set consisting of 10 DMUs, each with three inputs and two outputs, is considered. The interval input-output data are presented in Table 1, while the input cost vector and output price vector are given by

$$C = (0.5, 0.1, 0.5)^T$$

and

$$P = (0.2, 1)^T$$

respectively.

Models (7)–(9) were solved using GAMS 8.0, a high-level mathematical programming system for solving linear and nonlinear optimization problems. The computational results indicate that the maximum attainable profit is

$$f(B^{**}) = 61.50,$$

which corresponds to DMU 6 (see Table 2). The optimal budget for this DMU is uniquely determined as

$$B_L^* = 5.50, \qquad B_U^* = 5.50.$$

Since

$$B_L^* = B_U^*,$$

the optimal budget is uniquely determined as

$$B^{**} = 5.50.$$

Accordingly, the corresponding maximum attainable profit equals 61.50, and budget congestion occurs whenever the available budget exceeds the optimal budget level, i.e.,

$$B > 5.50.$$

Based on Theorems 3 and 4, the following observations can be made.

(i) If $B<5.50$, the optimal objective value $f(B)$ increases as the available budget increases.

(ii) If $B=5.50$, the DMU achieves the maximum attainable profit, $f(B)=61.50$.

(iii) If $B>5.50$, additional budget does not improve the economic performance of the DMU. Consequently, the DMU experiences budget congestion, since the additional budget provides no further improvement in the optimal production system.

These results demonstrate that the proposed interval OSD DEA model is capable of identifying both the economically optimal budget and the corresponding optimal production plan. Furthermore, the model provides a practical mechanism for detecting budget congestion under interval uncertainty, thereby supporting more efficient resource allocation and managerial decision-making.

To assess the robustness of the proposed approach, a brief sensitivity analysis was conducted by varying the width of the input and output intervals for DMU 6. Specifically, the interval widths were increased and decreased by 10% and 20%, and the corresponding optimal budget B^{**} and maximum profit $f(B^{**})$ were recomputed. The results, summarized in Table 3, show that the optimal budget and maximum profit remain stable under moderate variations in interval widths, indicating that the proposed model is robust to small changes in the interval bounds. However, large variations in interval widths (≥ 20%) can lead to changes in the optimal system design, highlighting the importance of accurate interval estimation in practice.

6. Discussion and Conclusion

The present study developed an interval-data Data Envelopment Analysis (DEA) framework for optimal system design (OSD), with particular emphasis on profit maximization, optimal budget determination, and the identification of budget congestion under uncertainty. The numerical findings demonstrated that the proposed model was capable of generating an economically optimal input-output configuration for each of the ten decision-making units (DMUs), while simultaneously incorporating uncertainty in benchmark production data. The resulting optimal designs differed substantially across the DMUs, indicating that economically desirable production configurations cannot be assumed to be uniform across units even when they operate within the same reference technology. The corresponding maximum attainable profits also varied considerably. In particular, DMU 6 achieved the highest maximum profit, equal to 61.50, whereas DMU 2 exhibited the lowest maximum profit. These results indicate that the proposed interval OSD approach differentiates DMUs according to their economic production potential while accounting explicitly for bounded uncertainty in their observed input-output information.

The observed heterogeneity in optimal input-output configurations is consistent with the fundamental logic of DEA, according to which each production unit is evaluated or designed with reference to the feasible production possibilities represented by observed peers rather than against a predetermined parametric technology. More importantly, the present findings extend conventional DEA analysis by demonstrating that economically desirable configurations can be identified even when benchmark production observations are represented by intervals rather than precise values. This finding is compatible with earlier interval DEA research showing that bounded input-output information can be incorporated into frontier analysis without eliminating the capacity of DEA to distinguish between alternative production possibilities [10]. Wang et al. established that efficiency outcomes could be meaningfully represented when production observations are known only within lower and upper bounds. The present findings extend this principle from efficiency assessment to prospective system design: instead of merely estimating an interval within which the efficiency of an existing DMU may fall, the proposed model uses interval-valued benchmark data to construct an economically optimal production target.

The results are also aligned with the target-setting application of interval DEA developed by Malekmohammadi et al., who demonstrated that interval information can support the determination of improvement targets rather than being confined to descriptive uncertainty assessment [11]. The current study advances this perspective by integrating target determination directly with a profit-maximizing objective and a financial resource constraint. Thus, the optimal input and output quantities obtained in the numerical example are not simply technical projections toward an efficient frontier. Rather, they constitute economically selected targets determined jointly by the empirical production technology, input costs, output prices, and the available budget. This distinction is significant because a technically efficient target is not necessarily the economically most advantageous target. The present results therefore support the broader movement in DEA from purely technical benchmarking toward economically informed system configuration.

The finding that the proposed optimal designs lie on an economically relevant efficient frontier is also consistent with the directional distance function literature. Directional distance functions provide a theoretical foundation for considering simultaneous output expansion and input contraction while preserving consistency with the production technology [1]. In the present framework, positive output prices and input costs similarly create a systematic incentive to increase desirable outputs and avoid unnecessary input consumption. Consequently, a dominated production configuration cannot constitute a profit-maximizing optimal design because an alternative configuration producing more output without additional input, or requiring less input without reducing output, would necessarily generate greater economic value. This relationship helps explain why the proposed interval OSD framework generates Pareto-efficient production targets while retaining an explicitly economic objective.

The study's results also reinforce previous findings concerning the distinction between technical performance and profitability. Research on Portuguese bank branches demonstrated that profitability differences can be decomposed into technical and allocative components, emphasizing that economic performance reflects not merely technological efficiency but also the appropriateness of resource allocation given economic conditions [2]. Subsequent work further developed decomposable measures of profit efficiency within DEA, showing that meaningful improvement strategies require consideration of the economic consequences associated with alternative input-output configurations [3]. Similarly, the present model determines the production configuration that maximizes the difference between generated revenue and incurred input cost. Therefore, the substantial variation in maximum profit across the ten DMUs should not be interpreted simply as variation in technical productivity; it reflects differences in economically attainable production possibilities conditional on the benchmark technology and budgetary structure.

The findings also accord with research demonstrating that profit-efficiency analysis can remain informative even when conventional economic information is incomplete. Kuosmanen et al. showed that profit efficiency can be evaluated using absolute and uniform shadow-price approaches, illustrating that economically meaningful frontier analysis is possible under imperfect price-information environments [4]. More recently, inverse DEA methods have been developed for cost and revenue efficiency where price information is unavailable, further illustrating the adaptability of economic DEA models to situations in which conventional economic data assumptions are not fully satisfied [5]. The current study addresses a different but related informational limitation: prices for the target system are assumed known, whereas production information for reference units is uncertain. The results show that this form of incomplete information can also be incorporated systematically without abandoning economic optimization.

A central result of the present study concerned optimal budget determination. For the numerical illustration associated with the maximum-profit production configuration, the optimal budget was identified as 5.50, at which the maximum attainable profit reached 61.50. When the available budget was below 5.50, increases in budget improved the optimal objective value; once the budget reached 5.50, however, the maximum attainable profit had been achieved. Additional financial resources above this threshold produced no further increase in the maximum profit under the original inequality-constrained model. This finding is theoretically important because it demonstrates that the relationship between financial resources and economic performance is not indefinitely increasing. Rather, there exists an economically sufficient budget beyond which additional financial capacity becomes redundant from the perspective of profit maximization.

This result directly supports the earlier OSD literature concerning optimal system design and budget determination. Wei and Chang showed that DEA can be transformed from an ex post performance-evaluation method into an ex ante system-design framework in which the economically desirable system and its associated budget are determined simultaneously [6]. Their profit-maximizing OSD formulation further established that input costs, output prices, and an available budget can be integrated with the DEA production possibility set to identify a production configuration that maximizes economic return [7]. The present findings extend these insights to interval-valued benchmark observations. Importantly, the existence of an identifiable optimal budget despite uncertainty in the production data indicates that the OSD logic remains applicable when the empirical production technology is described through bounded rather than exact observations.

The concave and non-decreasing relationship between the available budget and maximum attainable profit before the economically optimal budget is reached can also be understood through parametric linear programming. Parametric linear programming allows the right-hand side of a resource constraint to vary systematically and enables the resulting changes in the optimal objective value to be examined across parameter ranges [8]. Within the present framework, treating the budget as a parameter makes it possible to identify the point at which additional financial resources cease to improve the optimal economic outcome. This provides a stronger managerial interpretation than solving the OSD model for only a single predetermined budget, because managers can distinguish between resource scarcity, an economically sufficient budget, and excessive budget availability.

The observed budget pattern is also consistent with Fang's analysis of optimal budgets in system-design DEA. Fang demonstrated in a series-network context that optimal system performance need not require unrestricted increases in budget and that determining the appropriate budget level is a distinct optimization problem [9]. The present study supports this principle under interval uncertainty and extends it to the concept of budget congestion. Specifically, when the available budget exceeded 5.50, additional budget did not increase maximum attainable profit. Therefore, additional financial resources beyond the identified threshold did not create incremental economic value. This result provides a direct operational interpretation of budget congestion within an interval OSD environment.

The budget-congestion finding has particularly important implications. In conventional production analysis, congestion generally refers to circumstances in which excessive quantities of inputs do not increase outputs and may adversely affect performance. The current model adapts this logic to financial resources. When the decision maker is permitted to leave part of the available budget unused, an amount above the optimal level does not necessarily reduce maximum profit because the redundant resources need not be consumed. However, when full budget expenditure is imposed, forcing the production system to consume funds beyond the economically optimal level restricts the feasible allocation structure and can result in an inferior economic configuration. Thus, the study demonstrates that an allocated budget and an economically productive budget should not be treated as equivalent concepts.

This interpretation is particularly relevant for organizations in which complete budget expenditure is institutionally encouraged or administratively required. The model demonstrates mathematically that maximizing expenditure is not equivalent to maximizing economic performance. Once the economically optimal production scale has been attained, compulsory allocation of additional resources may generate unnecessary input consumption without producing sufficient additional output revenue. The result therefore contributes to the system-design literature by formalizing a financial counterpart of production congestion and identifying a measurable threshold beyond which further spending becomes economically ineffective. This conclusion extends earlier optimal-budget research [6, 9] by demonstrating that the phenomenon remains meaningful when the underlying production observations themselves are uncertain.

The present findings further demonstrate why uncertainty should be incorporated explicitly into economic DEA models. Robust DEA studies have shown that efficiency assessments based exclusively on deterministic observations can be sensitive to uncertainty in the underlying data. Data-driven robust DEA models have consequently been proposed to generate performance measures that better reflect uncertain operational environments [13]. Similarly, robust DEA with variable budgeted uncertainty provides mechanisms for controlling the extent of uncertainty considered within the optimization process [14]. Although the present study adopts an interval-data rather than a robust optimization formulation, the findings are consistent with the broader conclusion emerging from this literature: uncertainty in production information can materially affect frontier construction, benchmarking, and resource-allocation decisions and therefore should be represented explicitly in the analytical model.

The sensitivity analysis reinforces this conclusion. When the widths of the input and output intervals were moderately varied, the optimal budget and maximum profit remained comparatively stable, indicating that the proposed model was not excessively sensitive to small changes in interval specifications. However, larger changes in interval widths could alter the optimal design. This finding is theoretically plausible because interval bounds define the production possibilities from which the model constructs its economically optimal benchmark. Narrow adjustments that do not materially alter the relevant frontier may leave the optimal solution unchanged, whereas sufficiently large changes can change the relative attractiveness of reference combinations and therefore modify the selected production plan. Generalized DEA research with imprecise data similarly emphasizes that the manner in which uncertainty is represented can influence resulting efficiency characterizations [12]. The present findings therefore highlight both the usefulness of interval modeling and the importance of establishing credible interval bounds.

The results may also be viewed alongside alternative approaches to uncertainty in DEA. Chance-constrained DEA has demonstrated that uncertain production conditions can be integrated with profit and scale considerations when probabilistic information is available [15]. Likewise, chance-constrained network DEA has been employed to address uncertainty in multi-stage production structures [16]. These approaches differ methodologically from the interval representation employed in the present study, but they support the same broader proposition: production planning and efficiency analysis should recognize uncertainty rather than imposing exact observations when the available information does not justify such precision. Interval modeling is particularly useful when reliable probability distributions cannot be specified but defensible lower and upper bounds are available.

The findings additionally provide a foundation for connecting the present static OSD formulation with emerging dynamic and network economic DEA research. Integrated dynamic interval DEA demonstrates that interval uncertainty can be accommodated across multiple periods and in settings involving complex data structures [17]. Moreover, dynamic network models using directional distance functions have shown how cost-effectiveness can be evaluated across interconnected production stages and periods [18]. Recent value-based dynamic network DEA further integrates cost and revenue efficiency with explicit target setting [19]. The present study complements these developments by demonstrating that interval uncertainty can be incorporated directly into profit-maximizing system design and optimal budget analysis. Collectively, these strands suggest a progression toward DEA models that simultaneously recognize uncertainty, internal production structure, temporal dependence, economic performance, and managerial target setting.

Overall, the results indicate that the proposed interval OSD DEA framework successfully integrates four analytically related but frequently separated issues: uncertain production information, economically optimal system design, optimal budget determination, and budget congestion. The numerical application demonstrates that the model can distinguish economically optimal production plans across DMUs, identify the maximum attainable profit, determine the minimum financially sufficient level associated with that optimum, and diagnose situations in which additional budget ceases to create economic value. These findings support previous developments in interval DEA, economic efficiency analysis, optimal system design, robust and uncertain DEA, and optimal budgeting while extending them within a unified linear programming framework. Most importantly, the results suggest that uncertainty-aware production planning should focus not on maximizing the quantity of resources consumed but on identifying the combination of inputs, outputs, and financial resources that produces the greatest attainable economic value.

Several limitations of the present study should be acknowledged. First, uncertainty was incorporated into the input and output observations of the reference DMUs through interval bounds, whereas input costs and output prices were assumed to be known precisely. In many real production environments, however, costs and selling prices may themselves fluctuate considerably, and simultaneous uncertainty in technological and economic parameters could affect optimal designs. Second, the proposed formulation represents an optimistic planning framework based on favorable interval bounds and therefore may not correspond to the risk preferences of highly conservative decision makers. Third, the model considers a static production structure and does not explicitly represent intertemporal dependencies, carry-over activities, internal network stages, or dynamic adjustment costs. Fourth, the numerical illustration contains only ten DMUs with three inputs and two outputs and was designed primarily to demonstrate the analytical properties of the proposed method rather than to establish sector-specific empirical conclusions. Finally, interval results depend on the credibility of the specified lower and upper bounds; poorly estimated intervals that are excessively narrow or broad can influence the resulting optimal production technology and associated budget recommendations.

Future research should extend the proposed framework in several directions. Models could simultaneously incorporate uncertainty in production quantities, input costs, output prices, and available budgets to represent a broader spectrum of real-world economic uncertainty. Conservative, optimistic, and intermediate formulations could be developed within a unified framework so that decision makers can select targets according to different levels of risk tolerance. The method could also be extended to dynamic, multi-period, network, and multi-stage production technologies in which resources and intermediate products are transferred across stages or periods. Incorporating variable returns to scale, undesirable outputs, environmental constraints, and integer or categorical production variables would further increase the applicability of the approach. Future studies could compare interval formulations with robust, stochastic, fuzzy, and chance-constrained alternatives to determine how different uncertainty representations influence optimal budgets and production targets. Large-scale empirical applications in banking, healthcare, education, manufacturing, logistics, energy, and public administration are also needed to assess the external validity, computational performance, and managerial usefulness of the proposed framework under realistic institutional conditions.

For practical application, managers should use the proposed framework as a resource-allocation and planning instrument rather than solely as an efficiency-ranking procedure. Organizations can estimate defensible lower and upper bounds for uncertain inputs and outputs from historical observations, forecast ranges, expert assessments, or operational records and use these intervals to generate economically meaningful production targets. Decision makers should distinguish carefully between the budget formally available to an organization and the budget actually required to achieve maximum attainable economic performance. When the model identifies an optimal budget below the allocated amount, unused financial resources should not automatically be interpreted as evidence of poor management; instead, managers should examine whether additional expenditure would produce measurable economic value. Organizations that currently impose full-budget-consumption requirements should consider incorporating marginal economic performance into expenditure decisions to avoid budget congestion. Finally, interval estimates should be reviewed periodically, and sensitivity analyses should accompany major resource-allocation decisions so that managers can determine whether recommended production targets remain stable when underlying uncertainty changes.

Authors' Contributions

Authors equally contributed to this article.

Ethical Considerations

All procedures performed in this study were under the ethical standards.

Acknowledgments

Authors thank all participants who participate in this study.

Conflict of Interest

The authors report no conflict of interest.

Funding/Financial Support

According to the authors, this article has no financial support.

References

[1] R. Färe and S. Grosskopf, "Theory and Application of Directional Distance Functions," *Journal of Productivity Analysis,* vol. 13, no. 2, pp. 93-103, 2000.

[2] M. C. A. S. Portela and E. Thanassoulis, "Profitability of a Sample of Portuguese Bank Branches and Its Decomposition into Technical and Allocative Components," *European Journal of Operational Research,* vol. 162, no. 3, pp. 850-866, 2005, doi: 10.1016/j.ejor.2003.10.018.

[3] M. C. A. S. Portela and E. Thanassoulis, "Developing a Decomposable Measure of Profit Efficiency Using DEA," *Journal of the Operational Research Society,* vol. 58, no. 4, pp. 481-490, 2007, doi: 10.1057/palgrave.jors.2602169.

[4] T. Kuosmanen, M. Kortelainen, T. Sipiläinen, and L. Cherchye, "Firm and Industry Level Profit Efficiency Analysis Using Absolute and Uniform Shadow Prices," *European Journal of Operational Research,* vol. 202, no. 2, pp. 584-594, 2010, doi: 10.1016/j.ejor.2009.06.010.

[5] M. Ghiyasi, N. Nasrabadi, and A. Emrouznejad, "Inverse DEA Based on Cost and Revenue Efficiency in the Absence of Price Information," *Expert Systems with Applications,* vol. 284, p. 128028, 2025, doi: 10.1016/j.eswa.2025.128028.

[6] Q. Wei and T. S. Chang, "Optimal Profit-Maximizing System Design Data Envelopment Analysis Models," *Computers & Industrial Engineering,* vol. 61, no. 4, pp. 1275-1284, 2011, doi: 10.1016/j.cie.2011.08.001.

[7] Q. Wei and T. S. Chang, "DEA-Type Models for Designing Optimal Systems and Determining Optimal Budgets," *Mathematical and Computer Modelling,* vol. 54, no. 11-12, pp. 2645-2658, 2011, doi: 10.1016/j.mcm.2011.06.035.

[8] S. I. Gass, *Linear Programming: Methods and Applications*, 5th ed. Dover Publications, 2010.

[9] L. Fang, "Optimal Budget for System Design Series-Network DEA Model," *Journal of the Operational Research Society*, vol. 65, no. 12, pp. 1781-1787, 2014, doi: 10.1057/jors.2013.153.

[10] Y. M. Wang, R. Greatbanks, and J. B. Yang, "Interval Efficiency Assessment Using Data Envelopment Analysis," *Fuzzy Sets and Systems*, vol. 153, no. 3, pp. 347-370, 2005, doi: 10.1016/j.fss.2004.12.011.

[11] N. Malekmohammadi, A. B. Jaafar, and M. Monsi, "Setting Targets with Interval Data Envelopment Analysis Models via Wang Method," *Sains Malaysiana*, vol. 39, no. 3, pp. 485-489, 2010.

[12] J. Wu, W. Xie, and Y. Sheng, "A Generalized DEA Model with Imprecise Data," *Journal of Industrial and Management Optimization*, vol. 20, no. 8, pp. 2617-2639, 2024, doi: 10.3934/jimo.2024018.

[13] S. Qu, C. Feng, S. Jiang, J. Wei, and Y. Xu, "Data-Driven Robust DEA Models for Measuring Operational Efficiency of Endowment Insurance System of Different Provinces in China," *Sustainability*, vol. 14, no. 16, p. 9954, 2022, doi: 10.3390/su14169954.

[14] A. Arabmaldar, A. Hatami-Marbini, D. Loske, M. Hammerschmidt, and M. Klumpp, "Robust Data Envelopment Analysis with Variable Budgeted Uncertainty," *European Journal of Operational Research*, vol. 315, no. 2, pp. 626-641, 2024, doi: 10.1016/j.ejor.2023.11.043.

[15] L. Parhizkar Miyandehi, A. Amirteimoori, S. Kordrostami, and M. Soufi, "Optimal Scale Sizes and Average-Profit Efficiency under Uncertainty: A Chance-Constrained DEA Approach," *PLOS ONE*, vol. 19, no. 6, p. e0295241, 2024, doi: 10.1371/journal.pone.0295241.

[16] S. Yang and X. Yang, "A New Uncertain Two-Stage Network DEA Model Based on Chance-Constrained Programming," *Journal of Industrial and Management Optimization*, vol. 21, no. 3, pp. 2032-2051, 2025, doi: 10.3934/jimo.2024161.

[17] P. Bansal and A. Mehra, "Integrated Dynamic Interval Data Envelopment Analysis in the Presence of Integer and Negative Data," *Journal of Industrial and Management Optimization*, vol. 18, no. 2, pp. 1339-1363, 2022, doi: 10.3934/jimo.2021023.

[18] R. Kaur and J. Puri, "Analysing Cost-Effectiveness in Dynamic Network DEA: A Directional Distance Function Approach," *Operational Research*, vol. 24, no. 59, 2024, doi: 10.1007/s12351-024-00859-0.

[19] R. Kaur and J. Puri, "Cost and Revenue Efficiency Analysis in Dynamic Network DEA with Target Setting: A Value-Based Approach," *Operational Research*, vol. 26, no. 1, pp. 1-31, 2026, doi: 10.1007/s12351-025-00992-4.